RANDOMIZED METHODS FOR UNDERDETERMINED LINEAR INVERSE PROBLEMS

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We present a method based on random column sampling for solving underdetermined linear inverse problems. Random sampling allows reducing the dimensionality of the inverse problem and acts as a preconditioner for classical regularization methods. We provide error estimates when the random sampling method is coupled with Tichonov regularization. The proposed method achieves high accuracy while keeping a low computational cost.

Keywords: *underdetermined inverse problem, random sampling, Tichonov regularization*

1. Introduction

Many real-world problems require solving an underdetermined inverse problem. For example, reconstructing brain activity maps from MEG/EEG data requires solving a severely underdetermined inverse problem that is ill-posed and ill-conditioned. In such cases, regularization techniques must be used to make the solution feasible. In recent years, randomized methods have proven to be very efficient in reducing the dimensionality of large linear algebra problems. Initially used to construct low-rank approximations of large matrices, randomized methods were later successfully applied to regression problems as well (see [1, 2] and references therein). We present a method based on random sampling to solve underdetermined linear inverse problems. We observe that random sampling allows reducing the dimensionality of the inverse problem, thus acting as a preconditioner of classical regularization methods.

2. The random sampling method

Consider the discrete ill-posed inverse problem

$$A\,x = b\,, \tag{1}$$

where $A \in \mathbf{R}^{m\times n}$, with $m \ll n$, is a ill-conditioned (possibly large) matrix, $x \in \mathbf{R}^n$ is the quantity to be reconstructed and $b \in \mathbf{R}^m$ are the measurements affected by noise. Let us introduce the sampling matrix $S \in \mathbf{R}^{n\times c}$, with $c \ll n$, which randomly samples c columns of A, i.e.,

$$A_S = A\,S \in \mathbf{R}^{m\times c}. \tag{2}$$

Assuming that the columns of A are normalized so that they have unitary ℓ_2-norm, we can sample the columns drawing from the uniform probability distribution. Thus, the random sampling method consists in solving the reduced problem

$$A_S\, x_S = b\,, \qquad x_S \in \mathbf{R}^c. \tag{3}$$

3. Randomized Tichonov regularization

Tichonov regularization is a classical regularization method widely used in several fields [3]. It consists in solving the minimization problem

$$\min(\|A\,x - b\|^2 + \alpha\|x\|^2), \tag{4}$$

where $\alpha > 0$ is the regularization parameter. Let $A = U\Sigma V$ be the singular value decomposition (SVD) of A. It is well-known that this problem has a unique minimizer that can be written as

$$x^\alpha = V(\Sigma\Sigma^T + \alpha I)^{-1}\Sigma^T U b. \tag{5}$$

In the randomized version of the Tichonov regularization we used the sampled matrix A_S instead of A, i.e.,

$$x_S^\alpha = V_S(\Sigma_S\Sigma_S^T + \alpha I_S)^{-1}\Sigma_S^T U_S b, \tag{6}$$

where $A_S = U_S\Sigma_S V_S$ is the SVD of A_S. In case of uniform sampling with replacement it can be shown [**?**] that for the relative error $\mathcal{E} = \|x - S\,x_S\|/\|x\|$ the following estimate holds

$$\mathcal{E} \leq \frac{1}{\alpha}\max(\frac{n}{c}\nu - 1, 1)\sqrt{\nu}\,\kappa(A^T A + \alpha I)\sigma_{max}(A^T A + \alpha I), \tag{7}$$

where ν denotes the maximum number of times a column is drawn. $\kappa(A)$ and $\sigma_{max}(A)$ denote the conditioning and the maximum singular value of a matrix A, respectively. The estimate shows that when $c \ll n$ the error decreases fast as c increases reaching a minimum approximately when $\frac{n}{c}\nu = 2$. Then, the error increases as ν.

4. Conclusion

We applied the random sampling method to solve the MEG/EEG inverse problem [4, 5, 6]. The method can be easily applied to different devices and can be integrated with other methodologies. Numerical experiments show that the method achieves high accuracy while keeping a low computational cost, making it suitable for real-time applications with portable devices, such as brain-computer interface training or neurofeedback rehabilitation.

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